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ELASTIČNO-PLASTIČNI PRELAZ U TANKOM OBRTNOM DISKU PROMENLJIVE GUSTINE SA UKLJUČCIMA

ELASTIC-PLASTIC TRANSITION IN A THIN ROTATING DISC HAVING VARIABLE DENSITY WITH INCLUSION

Originalni naučni rad / Original scientific paper
UDK /UDC: 539.319
Rad primljen / Paper received: 5.2.2009.

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Ključne reči
- elastično
- plastično
- tečenje
- prelazni napon
- uključak
- obrtni disk
- gustina

Keywords
- elastic
- plastic
- yielding
- transitional stress
- inclusion
- rotating disc
- density

Izvod

Elasto-plastični prelaz u tankom obrtnom disku promenljive gustine sa uključcima je proučavan primenom teorije prelaza po Setu. Rezultati su diskutovani i prikazani grafički. Uočeno je da do tečenja kod obrtnog diska sa uključcima, izrađenog od stišljivog materijala dolazi na unutrašnjoj površini pri manjoj ugaonoj brzini u poređenju sa diskom od nestišljivog materijala, dok je potreban veći procent povećanja ugaone brzine da dođe do potpune plastičnosti. Radijalni naponi su veći na otvoru obrtnog diska od nestišljivog materijala. To znači da postoji tendencija loma na otvoru kod obrtnog diska promenljive gustine izrađenog od stišljivog materijala. Pod uticajem promenljive gustine povećava se veličina radijalnog i obimskog napona na unutrašnjoj površini za stanje potpune plastičnosti.

Abstract

Elastic-plastic transition in a thin rotating disc having variable density with inclusion has been studied by using Seth's transition theory. Results have been discussed and depicted graphically. It has been observed that the rotating disc with inclusion and made of compressible material yields at the internal surface at a lesser angular speed as compared to a disc made of incompressible material whereas it requires a highly percentage increase in angular speed to become fully plastic. Radial stresses are higher at the bore for rotating disc made of incompressible material. It means that rotating disc having variable density and made of compressible material have a tendency to fracture at the bore. Effect of variable density also increases the values of radial and circumferential stresses at the internal surface for fully-plastic state.

UVOD

Obrtni diskovi imaju široku primenu u tehnici, kao što su zupčanici velikih brzina, turbinski rotori, kompresori, zamajci i pogonski diskovi kompjutera. Primena obrtnih diskova u mašinama i konstrukcijama otvorila je brojne probleme u području mehanike čvrstih tela. Tanke obrtne ravne diskove izrađene od izotropnog materijala su detaljno analizirali Timošenko i Gudije, /1/, u elastičnom području tretirajući ih kao problem ravnog stanja napona. Čakrabarti /2/ i Hejman /3/ su rešili problem u plastičnom području pomoću uslova tečenja Treska. Oni su prvo uveli smenu $|T_{qq}|_{\max} = Y$ (konstanta u elastičnom rešenju) da bi odredili ugaonu brzinu za potpuno plastično stanje. Njihovo rešenje ne uvodi uslov ravnog stanja napona, što znači da mogu da se dobiju isti naponi i ugaone brzine uz zahtev da disk bude potpuno plastičan, bez korišćenja uslova ravnog stanja napona (tj. $T_{zz} = 0$).

INTRODUCTION

Rotating discs have a wide range of applications in engineering, such as high speed gears, turbine rotors, compressors, flywheel and computer's disc drive. The use of rotating disc in machinery and structures has opened many problems in domain of solid mechanics. Thin rotating flat discs made of isotropic materials are analysed extensively by Timoshenko and Goodier /1/ in the elastic range and treated as a plane stress problem. Chakrabarty /2/ and Heyman /3/ have solved the problem for the plastic range with the help of Tresca's yield condition. They first substituted $|T_{qq}|_{\max} = Y$ (a constant in the elastic solution) to get the angular velocity for fully plastic state. Their solution does not involve the plane stress condition, that is to say, we can obtain the same stresses and angular velocity required by the disc to become fully plastic without using the plane stress condition (i.e. $T_{zz} = 0$).

Uzimanjem u obzir uslova ravnog stanja napona u ovom problemu, Gupta i Šukla /4/ su dobili drugačije rešenje za stanje pune plastičnosti primenom teorije prelaza Seta. Redi i Srinat /5/ su istraživali uticaj gustine materijala na napone i pomeranja obrtnog ortotropnog kružnog diska. Pokazalo se da postojanje gradijenta gustine u obrtnom disku znatno utiče na napone i pomeranja. Analitičko rešenje za elastično-perfektno plastične obrtne diskove konstantne debljine i gustine je proučavao Gamer, /6/, primenom uslova tečenja Treska. Gamer je takođe proučavao analitička rešenja za takve diskove od materijala koji linearno deformacijski ojačavaju koristeći isti uslov tečenja. Given, /7/, je proširio taj rad na obrtne diskove sa funkcijom debljine i funkcijom gustine, i dobio njihova analitička rešenja uz primenu istog ponašanja materijala i uslova tečenja. Kako primena zavisnosti napona linearno deformacijskog ojačavanja i plastične deformacije i uslova tečenja Treska može dovesti do rešenja u zatvorenom obliku, oni su to koristili za proučavanje ponašanja vlaknastih kompozita pri termičkom i termo-mehaničkom opterećenju, /8/. Međutim, mnogi materijali pokazuju nelinearno ponašanje pri deformacijskom ojačavanju. Ovo ne može dovoljno dobro da se opiše korišćenjem linearne zavisnosti napona deformacijskog ojačavanja i plastične deformacije. Čak i kod malih deformacija, gde se očigledno nelinearnost pojavljuje u plastičnom području u blizini tačke tečenja na krivoj napon-deformacija, pogodno je da se koriste zavisnosti nelinearnosti napona deformacijskog ojačavanja i deformacije ili napona i plastične deformacije da bi se približili toj nelinearnosti. Da bi se to postiglo, zavisnost napon–plastična deformacija oblika polinoma je predložena za diskove od materijala koji deformacijski ojačava nelinearno i primenjena za rešavanje elastično-plastičnog problema nelinearnog deformacijskog ojačavanja obrtnih čvrstih diskova konstantne debljine i konstantne gustine, /9, 10/. Perfektna i idealna plastičnost su dve ekstremne karakteristike sa oštrim linijama, što fizički nije moguće. Teorija prelaza Seta, /11/, ne zahteva bilo kakve pretpostavke kao uslov tečenja, uslov nestišljivosti, i tako postavlja i rešava opštiji problem, a time mogu da se obrade i problemi u kojima se pretpostavljaju gornji uslovi. On koristi koncept mere generalizovane deformacije i asimptotsko rešenje u kritičnim ili prelaznim tačkama diferencijalnih jednačina za polje deformacija, što je uspešno korišćeno u velikom broju problema pri plastičnosti i puzanju, /12-21/.

Set, /12/, je definisao generalizovanu glavnu deformaciju

Taking into account the plane stress condition in the analysis of this problem, Gupta and Shukla /4/ obtained a different solution for the fully plastic state by using Seth's transition theory. Reddy and Srinath /5/ investigated the influence of material density on the stresses and displacements of a rotating orthotropic circular disc. It has been shown that the existence of a density gradient in a rotating disc influences the stresses and displacements significantly. The analytical solution of elastic-perfectly plastic rotating disks of constant thickness and density was studied by Gamer, /6/, using Tresca's yield condition. Gamer also studies the analytical solutions of such disc with a linear strain-hardening material behaviour using the same yield condition. Güven, /7/, extended this work of rotating discs of thickness function and density function, and obtained their analytical solution using the same material behaviour and yield condition. As the application of the linear strain-hardening stress-plastic strain relation and Tresca's yield condition can lead to a closed-form solution, they were also applied to study the behaviour of fibrous composites under thermal and thermo mechanical loading, /8/. However, many materials exhibit nonlinear strain-hardening behaviour. This cannot be described well enough using the linear strain-hardening stress-plastic strain relation. Even for a small deformation, as very obvious, nonlinearity occurs in the plastic region close to the yield point of stress-strain curves, it is also valuable to use nonlinear strain-hardening stress-strain or stress-plastic strain relations to approach this nonlinearity. To do this, a polynomial stress-plastic strain relation and a polynomial stress-strain relation have been proposed for discs of nonlinear strain-hardening materials and applied to solve nonlinear strain-hardening elastic-plastic problems of rotating solid discs with a constant thickness and constant density, /9, 10/. Perfect and ideal plasticity are two extreme properties of a sharp line which is not physically possible. Seth's transition theory, /11/, does not require any assumptions like an yield condition, incompressibility condition, and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at critical or turning points of the differential equations defining the deformed field and were successfully applied to a large number of problems in plasticity and creep, /12-21/.

Seth, /12/, has defined the generalized principal strain as

$$e_{ii} = \int_0^{e_{ii}^A} \left[1-2e_{ii}^A\right]^{\frac{n}{2}-1} de_{ii}^A = \frac{1}{n}\left[1-\left(1-2e_{ii}^A\right)^{\frac{n}{2}}\right],\ i = 1; 2; 3 \tag{1}$$

gde je n mera, a e_{ii}^A komponenta konačne deformacije po Almansiju. Za $n = -2; -1; 0; 1; 2$ ona daje Koši, Grin Henki, Svejnger i Almansi mere, respektivno.

U ovom radu je teorijom prelaza Seta proučavan elastično-plastični prelaz u tankom obrtnom disku promenljive gustine sa rukavcem. Gustina diska se menja po zakonu

where n is the measure and e_{ii}^A is the Almansi finite strain components. For $n = -2; -1; 0; 1; 2$ it gives Cauchy, Green Hencky, Swainger and Almansi measures, respectively.

Here, elastic-plastic transition in the thin rotating disc of variable density with shaft is investigated by Seth's transition theory. The density of the disc varies in the form

$$\rho = \rho_0 (r/b)^{-m} \tag{2}$$

Ovde je ρ_0 gustina za $\rho = b$, a m je parametar gustine.

Here ρ_0 is the density at $\rho = b$, m is the density parameter.

OSNOVNE JEDNAČINE

Razmatra se tanki disk promenljive gustine sa središnjim otvorom poluprečnika a i spoljnjeg poluprečnika b. Prstenasti disk je postavljen na kruti rukavac. Disk se obrće ugaonom brzinom ω postepeno rastuće veličine oko ose upravne na ravan diska i prolazi kroz središte, sl. 1. Pretpostavljena je konstantna debljina diska, dovoljno mala da se uspostavi ravno stanje napona, tako da je aksijalni napon T_{zz} jednak nuli. Komponente pomeranja u cilindrično polarnim koordinatama su date u obliku /12/,

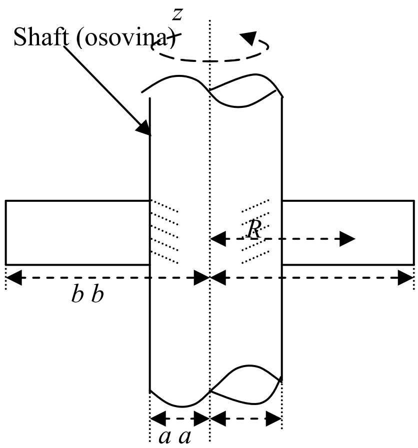
GOVERNING EQUATIONS

We consider a thin disc of variable density with central bore of radius a and external radius b. The annular disc is mounted on a rigid shaft. The disc is rotating with angular speed ω of gradually increasing magnitude about an axis perpendicular to its plane passing through the centre, Fig. 1. The thickness of disc is assumed to be constant and sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress T_{zz} is zero. The displacement components in cylindrical polar co-ordinates are given by /12/,

$$u = r(1-\beta) \qquad v = 0 \qquad w = dz \tag{3}$$

gde je β funkcija samo od $r = \sqrt{x^2 + y^2}$, a d je konstanta.

where β is a function of $r = \sqrt{x^2 + y^2}$ only, d is a constant.



Slika 1. Geometrija obrtnog diska
Figure 1. Geometry of rotating disc.

Komponente konačne deformacije, prema Setu, /12/, su

The finite strain components are given by Seth, /12/, as

$$e_{rr}^A \equiv \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right]; \; e_{rr}^A \equiv \frac{\partial u}{\partial r} - \frac{1}{2}\left(\frac{\partial u}{\partial r}\right)^2 = \frac{1}{2}\left[1-(r\beta'+\beta)^2\right]; \; e_{rr}^A \equiv \frac{\partial w}{\partial z} - \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 = \frac{1}{2}\left[1-(1-d)^2\right] \tag{4}$$

$$e_{r\theta}^A = e_{\theta z}^A = e_{zr}^A = 0 \tag{4a}$$

U jed. (4) i narednim jednačinama je $\beta' = d\beta/dr$.

Zamenom jed. (4) u jed. (1), generalizovane komponente deformacije su

In Eq. (4) and the following equations $\beta' = d\beta/dr$.

Substituting Eq. (4) in Eq. (1), the generalized components of strain are

$$e_{rr} = \frac{1}{n}\left[1-(r\beta'+\beta)^n\right]; \quad e_{\theta\theta} = \frac{1}{n}\left[1-\beta^n\right]; \quad e_{zz} = \frac{1}{n}\left[1-(1-d)^n\right]; \quad e_{r\theta} = e_{\theta z} = e_{zr} = 0 \tag{5}$$

Zavisnost napon–deformacija za izotropni materijal je, /23/,

The stress–strain relations for isotropic material are, /23/, given by

$$T_{ij} = \lambda \delta_{ij} I_1 + 2\mu e_{ij}, \; (i, j = 1, 2, 3) \tag{6}$$

gde su T_{ij} i e_{ij} komponente napona i deformacije, λ i μ su Lameove konstante, $I_1 = e_{kk}$ je prva invarijanta deformacije, a δ_{ij} Kronekerov delta simbol.

Jednačina (6) za ovaj problem prelazi u

where T_{ij} and e_{ij} are the stress and strain components, λ and μ are Lame's constants and $I_1 = e_{kk}$ is the first strain invariant, and δ_{ij} is the Kronecker's delta.

Equation (6) for this problem becomes

$$T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right] + 2\mu e_{rr}; \; T_{rr} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right] + 2\mu e_{rr}; \; T_{\theta\theta} = \frac{2\lambda\mu}{\lambda+2\mu}\left[e_{rr}+e_{\theta\theta}\right] + 2e_{\theta\theta}; \; T_{r\theta} = T_{\theta z} = T_{zr} = T_{zz} = 0 \tag{7}$$

Zamenom jed. (4) u jed. (6) komponente deformacije u zavisnosti od napona se dobijaju u obliku, /14/,

Substituting Eq. (4) in Eq. (6), the strain components in term of stresses are obtained as, /14/,

$$e_{rr}=\frac{1}{2}\left[1-(r\beta'+\beta)^2\right]=\frac{1}{E}\left[T_{rr}-\left(\frac{1-C}{2-C}\right)T_{\theta\theta}\right];\quad e_{\theta\theta}=\frac{1}{2}\left[1-\beta^2\right]=\frac{1}{E}\left[T_{\theta\theta}-\left(\frac{1-C}{2-C}\right)T_{rr}\right];$$
$$e_{zz}=\frac{1}{2}\left[1-(1-d)^2\right]=-\frac{(1-C)}{E(2-C)}\left[T_{rr}-T_{\theta\theta}\right];\quad e_{r\theta}=e_{\theta z}=e_{zr}=0 \tag{8}$$

gde je E modul elastičnosti i C faktor stišljivosti materijala, zavisan od Lameovih konstanti, koje imaju oblik

where E is the Young's modulus and C is compressibility factor of the material in term of Lame's constant, given by

$$E=\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)};\quad C=\frac{2\mu}{\lambda+2\mu}.$$

Zamenom jed. (5) u jed. (7), naponi se dobijaju u vidu

Substituting Eq. (5) in Eq. (7), we get the stresses as

$$T_{rr}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{1-C+(2-C)\left(\frac{r\beta'}{\beta}+1\right)^n\right\}\right];$$
$$T_{\theta\theta}=\frac{2\mu}{n}\left[3-2C-\beta^n\left\{2-C+(1-C)\left(\frac{r\beta'}{\beta}+1\right)^n\right\}\right];\quad T_{r\theta}=T_{\theta z}=T_{zr}=T_{zz}=0 \tag{9}$$

Sve jednačine ravnoteže su zadovoljene izuzev

Equations of equilibrium are all satisfied except

$$\frac{d}{dr}(rT_{rr})-T_{\theta\theta}+\rho\omega^2r^2=0 \tag{10}$$

gde je ρ gustina materijala diska. Unošenjem jed. (8) u jed. (9) dobija se nelinearna diferencijalna jednačina po β kao

where ρ is the density of disc material. Using Eq. (8) in Eq. (9), a non-linear differential equation in β is obtained as

$$(2-C)n\beta^{n+1}P(P+1)^{n-1}\frac{dP}{d\beta}=\frac{n\rho\omega^2r^2}{2\mu}+\beta^n\left[1-(P+1)^n-nP\left\{1-C+(2-C)(P+1)^n\right\}\right] \tag{11}$$

gde je $r\beta'=\beta P$ (P je funkcija od β, a β je funkcija od r).

Prelazi ili prelazne tačke za β u jed. (11) su $P\to-1$ i $P\to\pm\infty$.

Granični uslovi su

where $r\beta'=\beta P$ (P is function of β and β is function of r).

Transition or turning points of β in Eq, (11) are $P\to-1$ and $P\to\pm\infty$.

The boundary conditions are

$$u=0 \text{ za (for) } r=a \text{ i (and) } T_{rr}=0 \text{ za (for) } r=b \tag{12}$$

REŠENJE PREKO GLAVNIH NAPONA

Za određivanje plastičnih napona prelazna funkcija se uzima preko glavnih napona (vidi Set /12, 13/ i Gupta i sar. /14-21/) u prelaznoj tački $P\to\pm\infty$. Prelazna funkcija R je usvojena u obliku

SOLUTION THROUGH THE PRINCIPAL STRESS

For finding the plastic stress, the transition function is taken through the principal stress (see Seth /12, 13/ and Gupta et al. /14-21/) at the transition point $P\to\pm\infty$. We take the transition function R as

$$R=\frac{n}{2\mu}T_{\theta\theta}=\left[(3-2C)-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right] \tag{13}$$

Logaritamskim diferenciranjem jed. (13) po r i korišćenjem jed. (11), dobija se

Taking the logarithmic differentiation of Eq. (13) with respect to r and using Eq. (11), we get

$$\frac{d(\log R)}{dr}=-\frac{\beta^n\left(\frac{1-C}{2-C}\right)\left[1-(P+1)^n-n(1-C)P+\frac{n\rho\omega^2r^2}{2\mu\beta^n}\right]+(2-C)nP\beta^n}{r\left[3-2C-\beta^n\left\{2-C+(1-C)(P+1)^n\right\}\right]} \tag{14}$$

Uzimajući asimptotsku vrednost jed. (14) za $P\to\pm\infty$ posle integracije se dobija

Taking the asymptotic value of Eq. (14) at $P\to\pm\infty$ and integrating, one obtains

$$R=A_1r^{\nu-1} \tag{15}$$

gde je A_1 konstanta integracije, a $\nu=(1-C)/(2-C)$ je koeficijent Poasona. Iz jed. (13) i (15) sledi

where A_1 is an integration constant and $\nu=(1-C)/(2-C)$ is the Poisson's ratio. From Eq. (13) and Eq. (15), we have

$$T_{\theta\theta}=\left(\frac{2\mu}{n}\right)A_1r^{\nu-1} \tag{16}$$

Zamenom jed. (16) u jed. (10) i primenom jed. (2), posle integracije, dobija se

Substituting Eq. (16) in Eq. (10) and using Eq. (2), then integrating, we get

$$T_{rr} = \frac{B_1}{r} + \frac{2\mu A_1 r^{\nu-1}}{n\nu} - \frac{\rho_0 \omega^2 b^m r^{2-m}}{(3-m)}; \quad m \neq 3 \tag{17}$$

gde je B_1 konstanta integracije.

where B_1 is a constant of integration.

Zamenom jed. (16) i jed. (17) u drugu od jed. (9), dobija se

Substituting Eqs. (16) and (17) in the second of Eq. (9), we get

$$\beta = \sqrt{1 - \frac{2\nu}{E}\left[\frac{\rho_0 \omega^2 b^m r^{2-m}}{(3-m)} - \frac{B_1}{r}\right]}; \quad m \neq 3 \tag{18}$$

Zamenom jed. (18) u jed. (3) dobija se

Substituting Eq. (18) in Eq. (3), we get

$$u = r - r\sqrt{1 - \frac{2\nu}{E}\left[\frac{\rho_0 \omega^2 b^m r^{2-m}}{(3-m)} - \frac{B_1}{r}\right]}; \quad m \neq 3 \tag{19}$$

gde je $E = \frac{2\mu(3-2C)}{2-C}$ modul elastičnosti.

where $E = \frac{2\mu(3-2C)}{2-C}$ is the Young's modulus.

Unošenjem graničnih uslova (12) u jed. (17) i jed. (19) se dobija

Using boundary condition (12) in Eqs. (17) and (19), we get

$$A_1 = \frac{\rho_0 \omega^2 b^m n\nu (b^{3-m} - a^{3-m})}{2\mu(3-m)b^\nu}, \; B_1 = \frac{\rho_0 \omega^2 b^m a^{3-m}}{(3-m)}; \text{ za (for) } \; m \neq 3 \tag{20}$$

Zamenom vrednosti A_1 i B_1 u jednačinama (16), (17), i (19), redom, dobijaju se prelazni naponi (21) i pomeranja (22) u obliku

Substituting values A_1 and B_1 in Eqs. (16), (17), and (19), respectively, we get the transitional stresses (21) and displacement (22) in the form

$$T_{\theta\theta} = \frac{\rho_0 \omega^2 b^m \nu (b^{3-m} - a^{3-m})}{(3-m)r}\left(\frac{r}{b}\right)^\nu; \; T_{rr} = \frac{\rho_0 \omega^2 b^m}{(3-m)r}\left[(b^{3-m} - a^{3-m})\left(\frac{r}{b}\right)^\nu - r^{3-m} + a^{3-m}\right] \tag{21}$$

$$u = r - r\sqrt{1 - \left(\frac{2\nu}{E}\right)\frac{\rho_0 \omega^2 b^m (r^{3-m} - a^{3-m})}{(3-m)r}} \quad \text{za (for) } \; m \neq 3. \tag{22}$$

Iz jednačina (21) i (22) se dobija

From Eqs (21) and (22), we get

$$T_{rr} - T_{\theta\theta} = \frac{\rho_0 \omega^2 b^m}{(3-m)r}\left[(b^{3-m} - a^{3-m})(1-\nu)\left(\frac{r}{b}\right)^\nu - r^{3-m} + a^{3-m}\right], \text{ za (for) } m \neq 3 \tag{23}$$

Iz jed. (23) se vidi da je $|T_{rr} - T_{\theta\theta}|$ maksimalno na unutrašnjoj površini (za $r = a$), pa će do tečenja doći na unutrašnjoj površini diska, i iz jed. (23) sledi

From Eq. (23), it is seen that $|T_{rr} - T_{\theta\theta}|$ is maximum at the internal surface ($r = a$), therefore yielding will take place at the internal surface of the disc and Eq. (23) gives

$$|T_{rr} - T_{\theta\theta}|_{r=a} = \left|\frac{\rho_0 \omega^2 b^m (b^{3-m} - a^{3-m})(1-\nu)}{(3-m)a}\left(\frac{a}{b}\right)^\nu\right| \equiv Y \text{ (na pr.) (say) za (for) } \; m \neq 3. \tag{24}$$

a ugaona brzina Ω_i^2 potrebna za početno tečenje je

and the angular speed Ω_i^2 necessary for initial yielding

$$\Omega_i^2 = \frac{\rho_0 \omega_i^2 b^2}{Y} = \left|\frac{(3-m)ab^2}{b^m(1-\nu)(b^{3-m} - a^{3-m})}\left(\frac{b}{a}\right)^\nu\right| \quad \text{i (and) } \; \omega_i = \frac{\Omega_i}{b}\sqrt{\frac{Y}{\rho_0}} \tag{25}$$

Za $m = 3$ jed. (24) prelazi u

For $m = 3$ Eq. (24) becomes

$$|T_{rr} - T_{\theta\theta}|_{r=a} = \left|-\rho_0 \omega^2 b^2 \log\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)^{\nu-1}\right| \equiv Y \text{ (na pr.) (say)} \tag{26}$$

Disk postaje potpuno plastičan ($\nu \to 1/2$) na spoljnjoj površini i jed. (23) prelazi u

The disc becomes fully plastic ($\nu \to 1/2$) at the external surface and Eq. (23) becomes

$$\left|T_{rr} - T_{\theta\theta}\right|_{r=b} = \left|\frac{\rho_0 \omega^2 b^m (b^{3-m} - a^{3-m})}{2(3-m)b}\right| \equiv Y \quad \text{za (for)} \quad m \neq 3 . \tag{27}$$

Ugaona brzina Ω_f^2 potrebna da disk bude potpuno plastičan je

Angular speed Ω_f^2 required for the disc to become fully plastic is given by

$$\Omega_f^2 = \frac{\rho \omega_f^2 b^2}{Y} = \left|\frac{2(3-m)}{\left[1-(a^{3-m}/b^{3-m})\right]}\right|; \quad \omega_f = \frac{\Omega_f}{b}\sqrt{\frac{Y^*}{\rho}} \tag{28}$$

Za $m = 3$ jed. (27) prelazi u

For $m = 3$ Eq. (27) becomes

$$\left|T_{rr} - T_{\theta\theta}\right|_{r=b} = \left|-\rho_0 \omega^2 b^2 \log\left(\frac{a}{b}\right)\right| \equiv Y \tag{29}$$

Uvode se sledeće bezdimenzionalne komponente

Following non-dimensional components are introduced

$$R = \frac{r}{b};\; R_0 = \frac{a}{b};\; \sigma_r = \frac{T_{rr}}{Y};\; \sigma_\theta = \frac{T_{\theta\theta}}{Y};\; \frac{Y}{E} = H;\; \Omega^2 = \frac{\rho_0 \omega^2 b^2}{Y};\; \bar{u} = \frac{u}{b}.$$

Elastično-plastični prelazni naponi, ugaona brzina i pomeranja iz jednačina (21), (22) i (25) u bezdimenzionalnom obliku postaju

Elastic-plastic transitional stresses, angular speed and displacement from Eqs. (21), (22) and (25) in non-dimensional form become

$$\sigma_\theta = \frac{\Omega_i^2 R^\nu \nu}{(3-m)R}(1-R_0^{3-m}); \quad \sigma_r = \frac{\Omega_i^2}{(3-m)R}\left[(1-R_0^{3-m})R^\nu - R^{3-m} + R_0^{3-m}\right] \tag{30}$$

$$\Omega_i^2 = \frac{(3-m)}{(1-\nu)(1-R_0^{3-m})} R_0^{1-\nu} \tag{31}$$

$$\bar{u} = R - R\sqrt{1 - H\left[\frac{\Omega_f^2}{(3-m)R}(R^{3-m} - R_0^{3-m})\right]} \tag{32}$$

Za $m = 3$ naponi, ugaona brzina i pomeranja iz jednačina (30), (31) i (32) prelaze u oblik

For $m = 3$ stresses, angular speed and displacement from Eqs. (30), (31) and (32) become

$$\sigma_\theta = \frac{\Omega_i^2 R^\nu \nu}{R}(-\log R_0); \quad \sigma_r = \frac{\Omega_i^2}{R}\left[\log R_0 (1-R^\nu) - \log R\right] \tag{33}$$

$$\Omega_i^2 = -\frac{1}{(1-\nu)\log R_0} R_0^{1-\nu} \tag{34}$$

$$\bar{u} = R - R\sqrt{1 - 2\nu H\left[\frac{\Omega_i^2}{R}(\log R_0 - \log R)\right]}. \tag{35}$$

Naponi, ugaona brzina i pomeranja za potpunu plastičnost ($\nu \to 1/2$) se dobijaju iz jednačina (30), (31) i (32) kao

Stresses, angular speed and displacement for fully-plastic state ($\nu \to 1/2$) are obtained from Eqs. (30), (31) and (32) as

$$\sigma_\theta = \frac{\Omega_f^2}{2R(3-m)}(1-R_0^{3-m})R^{1/2}; \quad \sigma_r = \frac{\Omega_f^2}{(3-m)R}\left[(1-R_0^{3-m})R^{1/2} - R^{3-m} + R_0^{3-m}\right] \tag{36}$$

$$\Omega_f^2 = \frac{2(3-m)}{(1-R_0^{3-m})}, \tag{37}$$

$$\bar{u} = R - R\sqrt{1 - H\left[\frac{\Omega_f^2}{(3-m)R}(R^{3-m} - R_0^{3-m})\right]}. \tag{38}$$

Za $m = 3$ naponi, ugaona brzina i pomeranje iz formula (33), (34) i (35) postaju

For $m = 3$ stresses, angular speed and displacement from equations (33), (34) and (35) become

$$\sigma_\theta = \frac{\Omega_f^2 R^{1/2}}{2R}(-\log R_0); \quad \sigma_r = \frac{\Omega_f^2}{R}\left[\log R_0(1-R^{1/2}) - \log R\right] \tag{39}$$

$$\Omega_f^2 = \frac{-2}{\log R_0} \tag{40}$$

$$\bar{u} = R - R\sqrt{1 - H\left[\frac{\Omega_f^2}{R}(\log R_0 - \log R)\right]} \tag{41}$$

Rešenje za disk konstantne gustine (m = 0)

Kada je $m = 0$ prelazni naponi, ugaona brzina i pomeranja iz jednačina (30), (31) i (32) dobijaju oblik

Solution for a disc having constant density (m = 0)

When $m = 0$, the transitional stresses, angular velocity and displacement from Eqs. (30), (31) and (32) become

$$\sigma_\theta = \frac{\Omega_i^2 R^\nu \nu}{3R}(1-R_0^3); \quad \sigma_r = \frac{\Omega_i^2}{3R}\left[(1-R_0^3)R^\nu - R^3 + R_0^3\right] \tag{42}$$

$$\Omega_i^2 = \frac{3}{(1-\nu)(1-R_0^3)} R_0^{1-\nu} \tag{43}$$

$$\bar{u} = R - R\sqrt{1 - 2\nu H\left[\frac{\Omega_i^2}{3R}(R^3 - R_0^3)\right]} \tag{44}$$

Za stanje pune plastičnosti, naponi, ugaona brzina i pomeranja iz jednačina (36), (37) i (38) dobijaju oblik

For fully plastic-state, stresses, angular velocity and displacement from Eqs. (36), (37) and (38) become

$$\sigma_\theta = \frac{\Omega_f^2}{2R(3-m)}(1-R_0^{3-m})R^{1/2}; \quad \sigma_r = \frac{\Omega_f^2}{(3-m)R}\left[(1-R_0^{3-m})R^{1/2} - R^{3-m} + R_0^{3-m}\right] \tag{45}$$

$$\Omega_f^2 = \frac{2(3-m)}{(1-R_0^{3-m})} \tag{46}$$

$$\bar{u} = R - R\sqrt{1 - H\left[\frac{\Omega_f^2}{(3-m)R}(R^{3-m} - R_0^{3-m})\right]} \tag{47}$$

Jednačine (42) do (47) su iste kao jednačine koje su dobili Gupta i Pankaj, /22/.

Equations (42) to (47) are the same as obtained by Gupta and Pankaj, /22/.

NUMERIČKI PRIMER I DISKUSIJA

Iz tab. 1 se vidi da obrtni disk od nestišljivog materijala (guma, /23/) sa uključcima, zahteva veću ugaonu brzinu za tečenje na unutrašnjoj površini u poređenju sa diskom od stišljivog materijala (bakar, mesing, čelik, /23/), a da je veća ugaona brzina potrebna sa porastom odnosa prečnika. Takođe može da se vidi da je za stišljivi materijal potrebno veće procentualno povećanje ugaone brzine da dođe do pune plastičnosi nego za nestišljivi materijal. Pod uticajem promene gustine u obrtnom disku sa uključcima za stišljivi/nestišljivi materijal ugaona brzina se smanjuje sa povećanjem gustine ($m = -2$; 0; 2).

Na sl. 2 i 3 nacrtane su krive napona i pomeranja za tanki obrtni disk promenljive gustine ($m = -2$; 0; 2) zavisno od odnosa prečnika $R = r/b$ na elastično-plastičnom prelazu i za stanje pune plastičnosti, respektivno. Može se videti da je radijalni napon najveći na obimu otvora obrtnog diska izrađenog od nestišljivog materijal u poređenju sa stišljivim materijalom. To znači da obrtni disk promenljive gustine od stišljivog materijala teži da se lomi na otvoru. Sa povećanjem promene gustine ($m = -2$; 0; 2) vidi se uticaj na radijalne i obimske napone, tj. sa porastom promene gustine

NUMERICAL ILLUSTRATION AND DISCUSSION

From Table 1 it is seen that the rotating disc of incompressible material (rubber, /23/) with inclusions requires higher angular speed to yield at the internal surface compared to disc of compressible material (copper, brass, steel, /23/) and a higher angular speed is required to yield with the increase in radii ratio. It can also be seen that for compressible material higher percentage increase in angular speed is required to become fully plastic compared to incompressible material. Under the effect of density variation in a thin rotating disc with inclusion for compressible/incompressible materials, angular speed decreases with the increased density ($m = -2$; 0; 2).

In Figs. 2 and 3, curves are drawn for stresses and displacement for a thin rotating disc with variable density ($m = -2$; 0; 2) with respect to radii ratio $R = r/b$ at the elastic-plastic transition and fully plastic state, in respect. It has been seen that radial stress is maximum at the bore for rotating disc of incompressible material as compared to compressible material. It means the rotating disc with variable density of compressible material has a tendency to fracture at the bore. With an increase in density variation ($m = -2$; 0; 2) the effect is obvious on radial and circumferential stresses, with density smanjuje se veličina radijalnih i obimskih napona na unutrašnjoj površini za prelazno stanje, dok se na sl. 3 može videti da promena gustine povećava veličine radijalnog i obimskog napona na unutrašnjem prečniku za potpuno plastično stanje.

variation increase it decreases the value of radial and circumferential stress at the internal surface for transitional state whereas from Fig. 3, it can be seen that density variation increases the value of radial and circumferential stresses at the internal surface for fully plastic state.

Tabela 1. Veličina ugaone brzine potrebne za početno tečenje Ω_i u potpuno plastično stanje Ω_f

Table 1. The value of angular speed required for initial yielding Ω_i and fully plastic state Ω_f.

	Density	Angular speed	Incompressible material, $\nu = 0.5$	Compressible material, $\nu = 0.333$	Compressible material, $\nu = 0.29$
			rubber	copper, brass	steel
	Gustina	Ugaona brzina	Nestišljivi materijal, $\nu = 0,5$	Stišljivi materijal, $\nu = 0,333$	Stišljivi materijal, $\nu = 0,29$
			guma	bakar, mesing	čelik
$0.5 \le R \le 1$	m = –2	Ω_i^2	7.299167	4.877114	4.443946
		Ω_f^2	10.32258	10.32258	10.32258
	m = 0	Ω_i^2	4.848732	3.239797	2.95205
		Ω_f^2	6.857143	6.857143	6.857143
	m = 2	Ω_i^2	2.828427	1.889881	1.722029
		Ω_f^2	4	4	4
	m = –2	P	18.92070594	45.48314306	52.40869419
	m = 0	P	18.92071274	45.48315265	52.40869649
	m = 2	P	18.92071412	45.48317359	52.40870228

gde je $P = \left[\sqrt{\Omega_f^2 / \Omega_i^2} - 1 \right] \times 100$ procentualni porast ugaone brzine od početnog tečenja do stanja potpune plastičnosti.

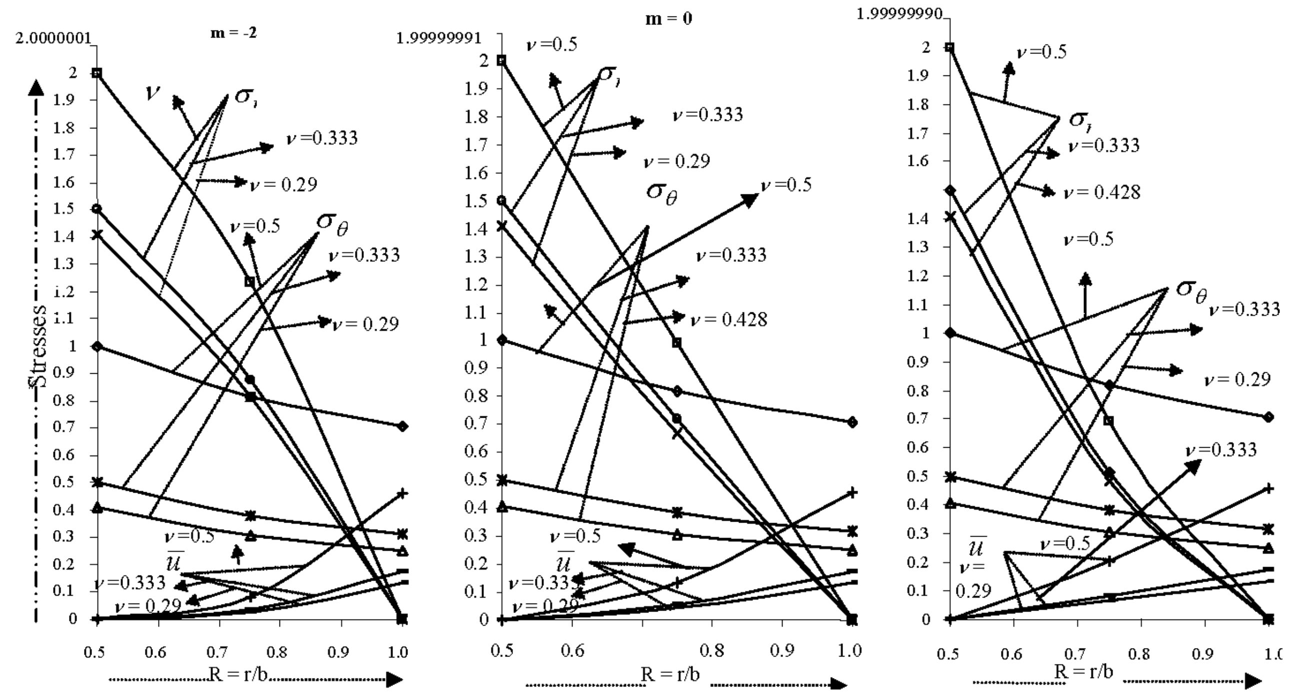
where $P = \left[\sqrt{\Omega_f^2 / \Omega_i^2} - 1 \right] \times 100$ is the percentage increase in angular speed from initial yielding to fully-plastic state.

ZAKLJUČAK

Uočeno je da obrtni disk sa uključcima, izrađen od stišljivog materijala plastično teče na unutrašnjoj površini pri manjoj ugaonoj brzini u poređenju sa nestišljivim materijalom, ali je potreban veći procent povećanja da se dostigne puna plastičnost. Radijalni napon je najveći na otvoru obrtnog diska od nestišljivog, u poređenju sa stišljivim materijalom. Promenljiva gustina povećava radijalni i obimski napon na unutrašnjoj površini za potpuno plastično stanje.

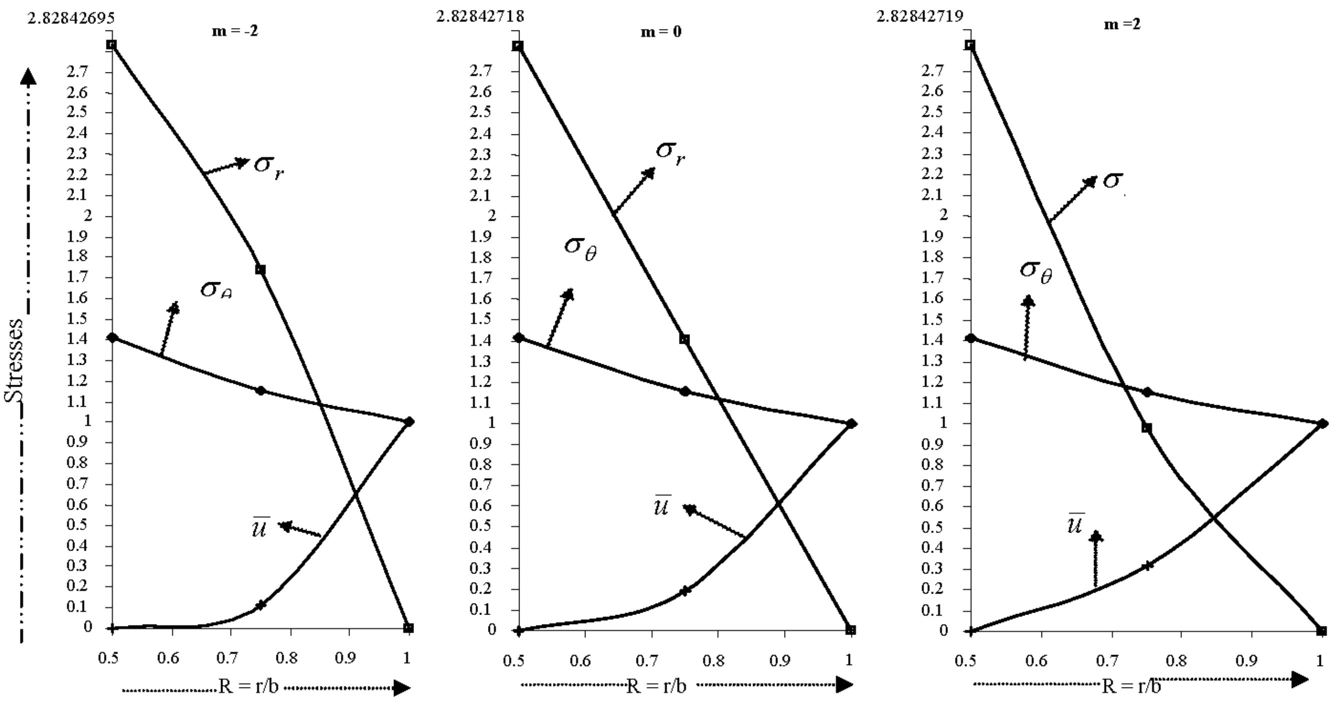
CONCLUSION

It has been seen that the rotating disc with inclusion made of compressible material yields at the internal surface at a lesser angular speed compared to incompressible material, but a higher percent increase in angular speed is required to become fully plastic. Radial stress is maximum at the bore on rotating disc of incompressible, compared to compressible material. Variable density increases radial and circumferential stress at the internal surface for fully-plastic state.



Slika 2. Naponi i pomeranja na elastičnom prelazu kod tankog obrtnog diska promenljive gustine (m = –2; 0; 2) zavisno od odnosa poluprečnika $R = r/b$

Figure 2. Elastic-plastic transition stresses and displacement in a thin rotating disc having variable density (m = –2; 0; 2) with respect to radii ratio $R = r/b$.



Slika 3. Naponi i pomeranja za punu plastičnost tankog obrtnog diska promenljive gustine (m = –2; 0; 2) zavisno od odnosa poluprečnika $R = r/b$

Figure 3. Fully-plastic stresses and displacement in a thin rotating disc having variable density (m = –2; 0; 2) with respect to radii ratio $R = r/b$.

ZAHVALNOST

Autor se zahvaljuje recenzentima za komentare, koji su doprineli znatnom poboljšanju rada.

ACKNOWLEDGMENT

The author is grateful to the referee for his critical comments, which led to a significant improvement of the paper.

LITERATURA – REFERENCES

1. Timoshenko, S.P. Goodier, J.N., Theory of elasticity, McGraw-Hill, New York, NY, 1951.
2. Chakrabarty, J., Theory of Plasticity, McGraw-Hill, New York, NY, 1987.
3. Heyman, J., *Plastic Design of Rotating Discs*, Proc. Inst. Mech. Engrs., 172 (1958), pp.531-546.
4. Gupta, S.K., Shukla, R.K., *Elastic-plastic transition in a thin rotating disc*, Ganita, 45 (1994), pp.78-85.
5. Reddy, T.Y., Srinath, H., *Elastic stresses in a rotating anisotropic annular disk of variable thickness and variable density*, Int. J. Mech. Sci., 16 (1974), pp.85-89.
6. Gamer, U., *Tresca's yield condition and the rotating disk*, Journal of Appl. Mechanics, Transactions of ASME, 50 (1983), pp.676-678.
7. Güven, U., *Elastic-plastic stress in a rotating annular disk of variable thickness and variable density*, International Journal of Mechanical Sciences, 34 (1992), pp.133-138.
8. You, L.H., Long, S.Y., Rohar, L., *Elastic-plastic stress field in a coated continuous fibrous composite subjected to thermomechanical loading*, Journal of Applied Mechanics, Transaction of ASME, 66 (3) (1999), pp.750-757.
9. You, L.H., Long, S.Y., *Effects of material properties of interfacial layer on stresses in fibrous composites subjected to thermal loading*, Composite Part A, 29(A) (1998), pp.1185-1192.
10. You, L.H., Zhang, J.J., *Elastic-plastic stresses in a rotating solid disk*, International Journal of Mechanical Sciences, 41 (1999), pp.269-282.
11. Seth, B.R., *Transition theory of elastic-plastic deformation, creep and relaxation*, Nature, 195 (1962), pp.896-897.
12. Seth, B.R., *Measure concept in mechanics*, Int. J. Non-linear Mech., 1 (1966), pp.35-40.
13. Seth, B.R., *Elastic-plastic transition in shells and tubes under pressure*, ZAMM, 43 (1963), pp.345-351.
14. Gupta, S.K., Sharma, S., Pathak, S., *Elastic-plastic transition in a thin rotating disc of variable density with edge loading*, Proc. National Acad. Sci. India, 70(A), I (2000), pp.75-86.
15. Gupta, S.K., Pankaj, T., *Creep transition in a thin rotating disc with rigid inclusion*, Defence Science Journal, 57 (2007), pp.185-195.
16. Gupta, S.K., Dharmani, R.L., Rana, V.D., *Creep transition in torsion*, Int. J. Non-linear Mechanics, 13 (1979), pp.303-309.
17. Gupta, S.K., Dharmani, R.L., *Creep transition in bending of rectangular plates*, Int. J Nonlinear Mech., 15 (1980), pp.147-154.
18. Gupta, S.K., Dharmani, R.L., *Creep transition in thick walled cylinder under internal pressure*, ZAMM, 59 (1979), pp.517-521.
19. Gupta, S.K., Pankaj, T., *Thermo elastic-plastic transition in a thin rotating disc with inclusion*, Thermal Science, 11 (2007), pp.103-118.
20. Gupta, S.K., Sharma, S., *Proceeding for 3rd International congress on Thermal Stresses*, Cracow, Poland, June 13-17 (1999), pp.345-348.
21. Gupta, S.K., *Thermo elastic-plastic transition of thick-walled rotating cylinder*, Proc. first Int. Symp. on Thermal Stresses and Related Topics, June 5-7 (1995) Japan, Hamamatsu.
22. Pankaj, T., PhD Thesis, Department of Mathematics, H.P. University Shimla, India, 2006, pp.43-54.
23. Sokolinikoff, I.S., Mathematical theory of elasticity, McGraw-Hill Book, Second Edition, New York, NY, 1950, pp.70-71.